

Lecture 14 Periodic Motion I

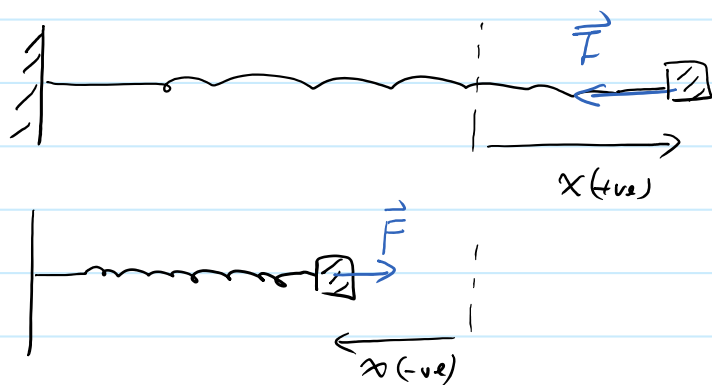
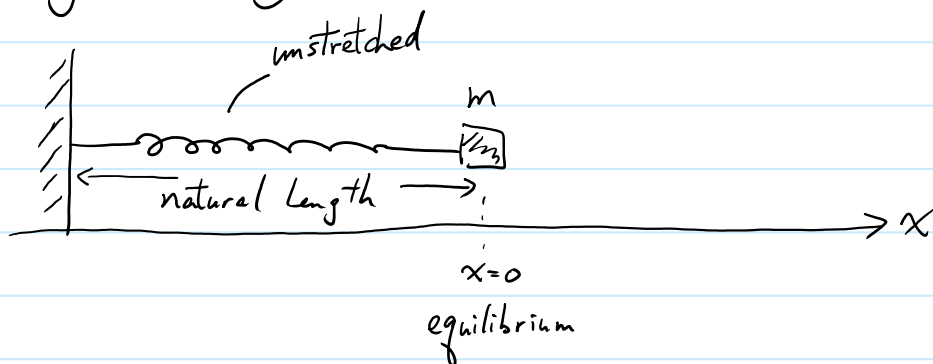
Periodic Motion = motion repeating in a fixed period of time.

e.g. planetary motion, clock, pendulum, vibration, wood floating on water surface, even chemical reactions!

Simple Harmonic Oscillation (oscillator) — SHM

- Restoring force is linearly proportional to displacement from the equilibrium position.

Spring-Mass system.



Hooke's Law

$$\vec{F} = -kx\hat{i}$$

or $F_x = -kx$.

restoring force: push the mass back to equilibrium.

Equation of motion: $\vec{F} = m\vec{a}$

$$-kx = F_x = ma_x = m \frac{d^2x}{dt^2}$$
$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$a_x = -\frac{k}{m}x \quad \text{or} \quad \ddot{x} = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

↑ Second Order Differential Equation of the form.

$$\ddot{x} = -\omega^2 x, \quad \omega^2 > 0 \quad \text{---} (*)$$

Simple harmonic motion (SHM)

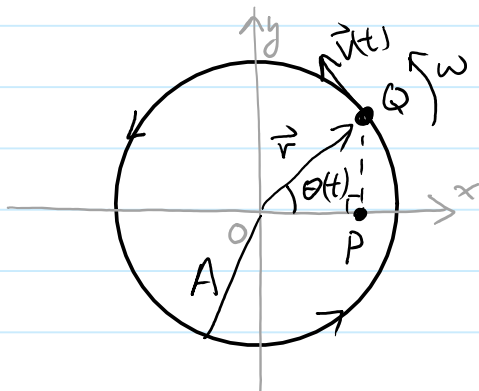
Solving (*) means to find a function $x(t)$ such that it satisfies (*).

How to find $x(t)$?

i.e. What $x(t)$ would have a 2nd derivative equal to a negative number times itself?

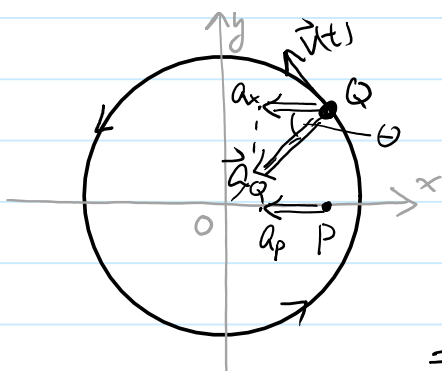
Solution of diff. eqn is unique.
i.e. once you find a $x(t)$ that satisfies (*), then this $x(t)$ is the only solution.

Consider a particle, Q, in uniform circular motion.



P is the projection of Q along x-axis.
i.e. x-component of $\vec{r}$

$$\text{we know } \vec{r} = \begin{cases} x(t) = A \cos(\theta(t)) = x_p \\ y(t) = A \sin(\theta(t)) \end{cases} \quad \text{where } \theta(t) = \omega t$$



for acceleration of Q: $\vec{a}_Q$

$$a_Q = \frac{v^2}{A} = A\omega^2, \quad v = A\omega$$

Acceleration of P: point to left.

$$a_p = a_x = -a_Q \cos \theta = -\omega^2 \underbrace{A \cos \theta}_{x_p}$$

⇒

$$\underline{a_p = \ddot{x}_p = -\omega^2 x_p}$$

Now we find a $x(t)$ such that $\ddot{x} = -\omega^2 x$

That is the position of the shadow of a uniform circular motion.

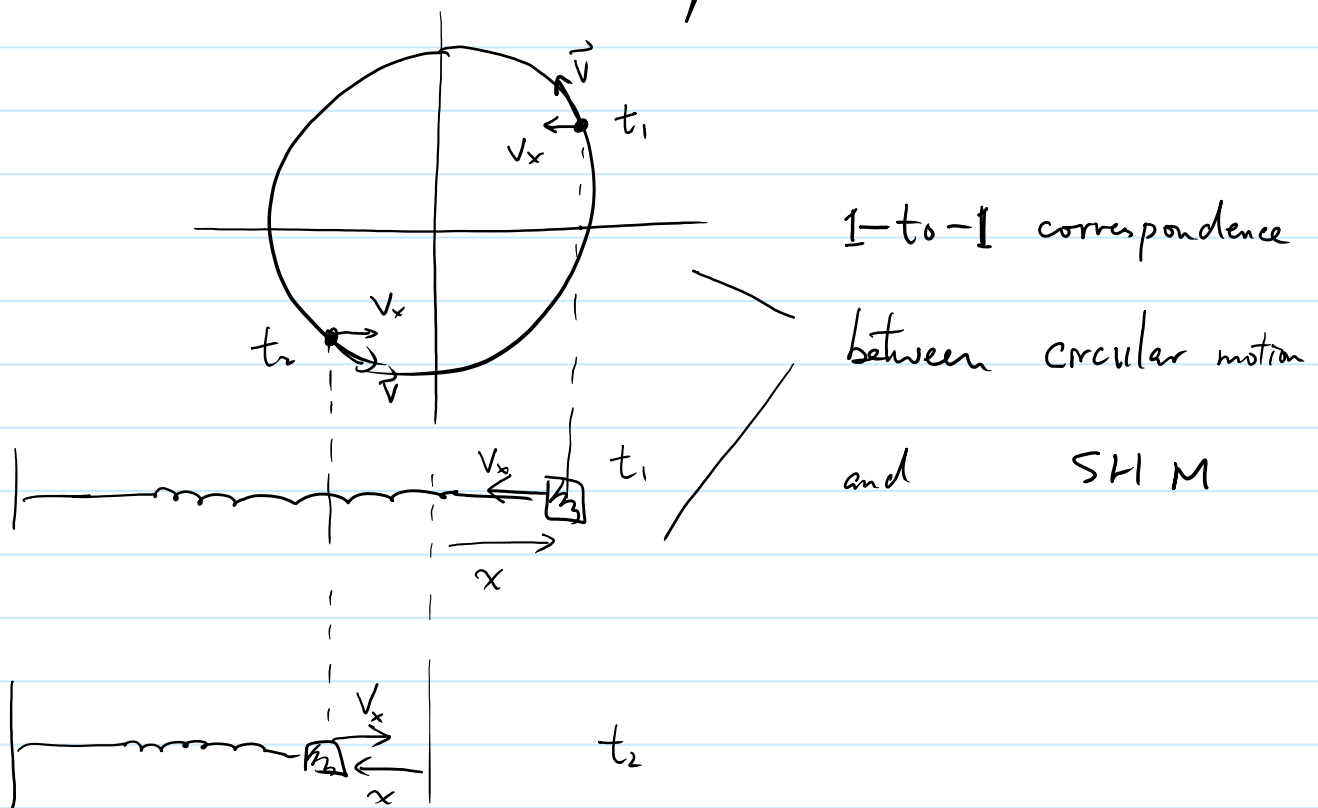
$$x(t) = A \cos \omega t$$

$$\theta(t) = \omega t$$

To describe SHM, one just needs to

consider a uniform circular motion with $R = A$ & $\omega = \frac{k}{m}$,

and take its horizontal component.



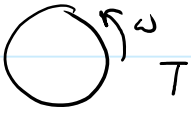
Compare $a_x = -\frac{k}{m}x$ to $a_x = -\omega^2 x$

$$\boxed{\omega^2 = \frac{k}{m}},$$

In circular motion

$$\omega = \frac{2\pi}{T}$$

$\uparrow$ angular speed. $\leftarrow$ period in circular motion.



Period in SHM = Period in circular motion = T

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

Frequency

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \# \text{ of cycles per unit of time.}$$

Angular frequency

$$\omega = 2\pi f. \quad \text{instead of angular velocity.}$$

General Solution

$$x(t) = A \cos(\omega t + \phi)$$

Amplitude $\underbrace{\hspace{1cm}}$ phase angle $\underbrace{\hspace{1cm}}$

(A, ϕ) are determined by initial condition. (x_0, v_0)

As a result:

$$v(t) = \dot{x}(t) = -\omega A \sin(\omega t + \phi) \equiv -V_{\max} \sin(\omega t + \phi)$$
$$a(t) = \ddot{x}(t) = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x(t)$$

ω depends on the parameter of the system.

$$\omega = \sqrt{\frac{k}{m}} \text{ for spring-mass system.}$$

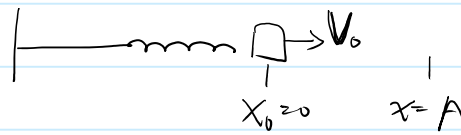
To determine A and ϕ for a given initial condition, (x_0, v_0)

$$\text{At } t=0, \quad x(t=0) = x_0, \quad v(t=0) = v_0$$

$$\Rightarrow \begin{cases} x_0 = x(0) = A \cos \phi \\ v_0 = \dot{x}(0) = -\omega A \sin \phi \end{cases}$$

$$\Rightarrow \begin{cases} x_0^2 + \frac{v_0^2}{\omega^2} = A^2 \cos^2 \phi + A^2 \sin^2 \phi = A^2 \\ \tan \phi = -\frac{v_0}{\omega x_0} \end{cases}$$

$$\begin{cases} A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \\ \phi = \begin{cases} \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right) & \text{for } x_0 \geq 0 \\ \tan^{-1}\left(\frac{-v_0}{\omega x_0}\right) + \pi & \text{for } x_0 < 0 \end{cases} \end{cases}$$

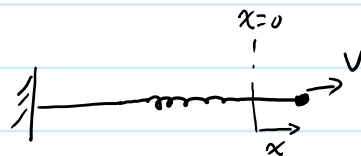
Example: at $t=0$ 

$$\Rightarrow \begin{cases} A = \frac{v_0}{\omega} \\ \phi = -\frac{\pi}{2} \end{cases} \Rightarrow x(t) = \frac{v_0}{\omega} \cos\left(\omega t - \frac{\pi}{2}\right)$$

Energy

$$E = K + U$$

$$K = \frac{1}{2} m v^2$$



$$U = \frac{1}{2} k x^2, \quad \omega^2 = \frac{k}{m}$$

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$\downarrow$$
$$= \frac{1}{2} k A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$E = \frac{1}{2} k A^2$$

$\uparrow$

U at $x=A$

where $K=0$

depends only on A .

$$\text{or } \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} m v_{\max}^2$$

$\uparrow$

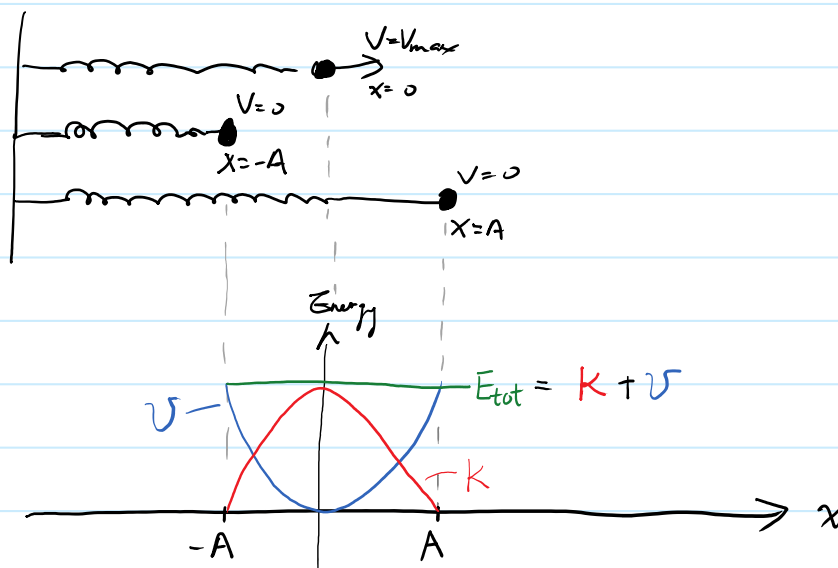
K at $x=0$

where $U=0$

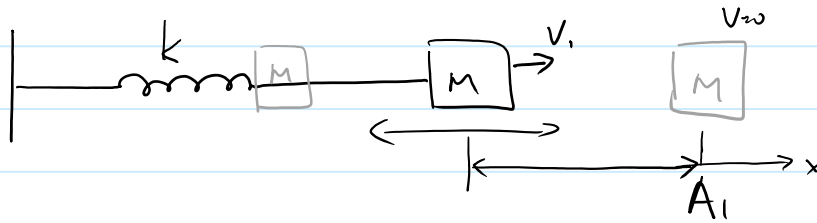
depends only on $v_{\max}$.

Since A and $v_{\max} = \omega A$ are constant,

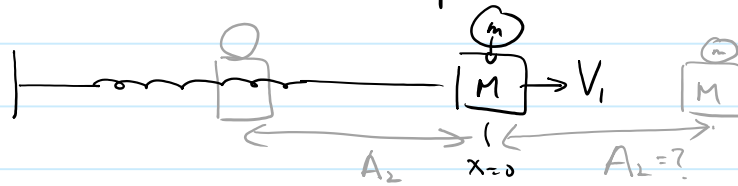
total mechanical energy in SHM is conserved.



Example



When the block is at $x=0$, a bubble gum of mass m is dropped and sticks on the block. Then they oscillate together. Find the new amplitude A_2 .



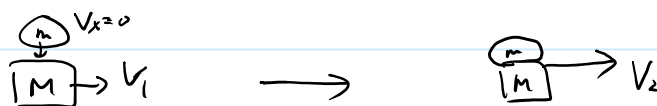
Before collision

By conservation of Energy

$$E = \text{at } x=0 \quad \frac{1}{2} M v_i^2 = \text{at } x=A \quad \frac{1}{2} k A^2$$

$$v_i = \sqrt{\frac{k}{M}} A_1$$

Right after collision



$$p_i = p_f$$

$$M v_i = (m+M) v_2$$

$$v_2 = \frac{M}{m+M} v_i = \frac{M}{m+M} \sqrt{\frac{k}{M}} A_1$$

Oscillation after collision

$$E = \frac{1}{2} (m+M) v_2^2 = \frac{1}{2} k A_2^2$$

$$\Rightarrow A_2 = \sqrt{\frac{(m+M)}{k}} v_2 = \sqrt{\frac{(m+M)}{k}} \cdot \frac{M}{(m+M)} \sqrt{\frac{k}{M}} A_1$$

$$A_2 = \sqrt{\frac{M}{m+M}} A_1 < A_1 \quad \text{is } \omega \text{ the same?}$$

still the same spring, k ,

What if the bubble gum is dropped when the block is at $x = A$?

Would $\begin{pmatrix} \text{amplitude} \\ \text{frequency} \\ v_{\max} \\ E_{\text{tot}} \end{pmatrix}$ changes?

Hint: Both objects has zero velocity when they stick.
 $\Rightarrow$ No relative velocity.



No kinetic friction $\Rightarrow$ No energy loss during collision.

$\Rightarrow$ Energy remains unchanged.